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A NOTE ON THE MAXIMUM NUMBER AND DENSITY OF DISTRIBUTION OF MEMBERS IN ELASTIC STRUCTURES OF MINIMUM WEIGHT UNDER MULTIPLE LOADING CONDITIONS

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Abstract—The problem of minimum weight design of elastic structures under multiple loading conditions is considered. It is shown that the problem can be expressed as a search for feasible deflection patterns coupled with repeated searches for structures of minimum weight for given stiffness. The latter is a Linear Programming Problem and implies an upper limit, different from that set by connectivity, on both the number and distribution density of elements present in the minimum weight design.

The problem of optimising elastic structures in the minimum-weight sense is one which is now receiving increasing attention, see e.g. [1]. Perhaps the largest area of effort is that of elastic structures under alternative sets of applied loads, when limits are placed on the stresses in the elements of the structure and the deflections of the nodes. Although some work (e.g. [2]) has been directed towards choosing the geometry of the structure, i.e. the number and positions of the nodes, the problem of choosing cross-sectional areas of members in a structure of fixed geometry is the best that can be hoped for in many cases.

This problem can, in general, be formulated as follows (using vector notation):

Minimise $\bar{w}^t\bar{A}$ 1(a)

Subject to:

$$\bar{S}_L \leqslant S_j(\bar{A}) \leqslant \bar{S}_u \quad 1(b)$$

$$\bar{V}_L \leqslant V_j(\bar{A}) \leqslant \bar{V}_u \quad 1(c)$$

$$\bar{A} \geqslant 0 \quad 1(d)$$

$$j = 1, 2, \ldots, M$$

where $\bar{w}$ and $\bar{A}$ are vectors of weights/unit cross-sectional area and actual cross-sectional area respectively, $\bar{S}_j$ and $\bar{V}_j$ are respectively values of stress failure criteria and deflection under the jth applied load set P_j. Let the structure be defined as a set of nodes of given coordinates, joined by an arbitrary number of members N. For any value of $\bar{A}$, the deflection can be computed from:

$$\bar{V}_j = \bar{K}^{-1}\bar{P}_j \quad (2)$$

where K, the stiffness matrix, is a linear function of $\bar{A}$. Then, $\bar{S}_j$ can be computed from $\bar{V}_j$ and the individual element stiffness matrices.

We will now consider an alternative formulation which sheds more light on the nature of the problem than formulation (1) in the particular case of elastic structures of fixed geometry. Note first that the stresses, being linearly dependent on the strains, are also linear functions of the $\bar{V}_j$ and so the S_j (which may simply be vectors of stresses, as in a pin-jointed structure: or nonlinear functions of stress, e.g. Von Mises Criteria) can be written as functions of $\bar{V}_j$. Thus, if the weight function can also be expressed as a function of $\bar{V}_j$, formulation (1), a problem in $\bar{A}$, could be replaced by a problem in $\bar{V}_j$. Let such a weight function be designated $W(\bar{V}_1, \ldots, \bar{V}_M)$, $(\equiv W(\bar{V}_j))$. Then (1) becomes:

Minimise $W(\bar{V}_j)$ 3(a)

Subject to:

$$\bar{S}_L \leqslant \bar{S}(\bar{V}_j) \leqslant \bar{S}_u \qquad 3(b)$$

$$\bar{V}_L \leqslant \bar{V}_j \leqslant \bar{V}_u \qquad 3(c)$$

$$j = 1, 2, 3, \ldots, M \qquad 3(d)$$

Where the search is confined to values of $\bar{V}_j$ for which

$$W(\bar{V}_j) > 0$$

Now consider how $W(\bar{V}_j)$ must be defined if a solution to (3) is to be identical with a solution to (1). Clearly, for this condition to hold, $W(\bar{V}_j)$ must be the minimum weight of a structure of the given geometry which will exhibit the deflections $\bar{V}_j$ under the loads P_j, at the same time, of course, satisfying the equilibrium and compatibility equations of the structure. Hence, $W(\bar{V}_j)$ is defined

$$W(\bar{V}_j) \equiv \{\text{Min } \bar{w}^t \bar{A}\} \qquad 4(a)$$

Subject to:

$$\bar{P}_j = \bar{K}(\bar{A})\bar{V}_j \qquad 4(b)$$

$$\bar{A} \geqslant 0 \qquad 4(c)$$

$$j = 1, 2, \ldots, M.$$

Now $\bar{K}$, the stiffness matrix, can be written

$$\bar{K} = \sum_{i=1}^{N} \bar{a}_i{}^t \bar{k}_i \bar{a}_i \qquad (5)$$

where $\bar{a}_i$ is a transformation matrix which is only a function of geometry; and $\bar{k}_i$ is an element stiffness matrix, linear in A_i. Since, in equation 4(b), the vectors $\bar{V}_j$ are given, the equation is in fact a set of linear equations in $\bar{A}$:

$$\bar{B}_j \bar{A} = \bar{P}_j \qquad 4'(b)$$

where

$$\bar{B}_j = [\bar{a}_1{}^t \bar{k}_1{}' \bar{a}_1 \bar{V}_j \mid a_2{}^t \bar{k}_2{}' a_2 \bar{V}_j \mid \cdots\cdots \mid \bar{a}_n{}^t \bar{k}_n{}' a_n \bar{V}_j]$$

and

$$\bar{k}_i{}' = \bar{k}_i / A_i, \text{ the constant part of } \bar{k}_i.$$

Let D be the number of degrees of freedom of the (supported) structure. Consider the equation 4′(b). Each vector $\bar{P}_j$ has D components, and so $\bar{B}_j$ has D rows. The vector $\bar{A}$ has N elements, so $\bar{B}$ has N columns. It follows that 4′(b) represents $M \times D$ equations in N unknowns. There are two cases to be considered.

(i) $M \times D \leqslant N$.

In this case, formulation (4) is clearly a Linear programming problem in $\bar{A}$.

(ii) $M \times D > N$.

Formulation (3)–(4) is then not strictly equivalent to formulation (1) because equations 4′(b) cannot be solved for arbitrary $\bar{V}_j$, i.e. $W(\bar{V}_1, \ldots, \bar{V}_M)$ is not defined for some set of values of $\bar{V}_j$; additional constraints would have to be included in (3) to ensure compatibility of 4′(b). However, if a solution to the problem exists, then the formulations are equivalent at the solution.

From these considerations, the following theorem can be stated:

Theorem

The maximum number of elements in an elastic structure of minimum weight for a prescribed geometry, subject to stress and deflection constraints under multiple alternative sets of applied loads, is equal to the product of the number of load cases and the number of degrees of freedom of the supported structure.

The theorem follows from the well-known Linear Programming result (e.g. [3]) which states that a linear program with M equality constraints and N variables, has, at the solution, at most M non-zero variables. Thus it is true in case (i), and is automatically satisfied in case (ii).

There is an interesting corollary to the theorem. Consider a minimum weight structure having deflections say $\bar{V}_j^*$. Any substructure S_1 of such a system can be considered in isolation, so long as no changes are made to S_1 which alter the deflections, i.e. the stiffness of S_1. Clearly, S_1 must be the structure of minimum weight for that stiffness, since, if it were not, a substructure of lower weight could be substituted without altering $\bar{V}_j^*$; since the stress and deflection constraints are functions of $\bar{V}_j$ alone, such a substructure would also be feasible. This violates the hypothesis that the initial overall structure is of minimum weight, and so the corollary can be stated:

Corollary 1

The limit set by the main theorem applies separately to every substructure within the total structure.

Thus, there is an upper limit on the density of distribution of members within a minimum-weight elastic structure. (It should perhaps be mentioned that corollary 1 does not imply that an optimum structure can be arrived at by optimising substructures in isolation, but merely that a structure thus designed would be subject to the same limits as a true optimum structure.)

To illustrate the implications of the theorem, consider the case of a pin-jointed frame, subject to one load case. Here, the theorem implies that the number of elements in the minimum-weight frame is equal to the number of equilibrium equations at the nodes. This could mean that the frame is statically determinate; or that it is redundant in some areas, and a mechanism (which happens to be stiff under the particular load set) in others. However, corollary 1 denies the possibility of the second case, and so we can state:

Corollary 2

The minimum weight pin-jointed elastic frame of prescribed geometry, under one load set and subject to stress constraints and/or deflection constraints, is statically determinate.

Corollary 2 is of course well known, at least in the case of either stress or deflection constraints; the reasoning above shows that it is simply a special case of a more general theorem which applies to a larger class of structure and loading requirements.

The limits derived in this note in fact refer to the optimisation of any elastic structure for which both weight and stiffness are linear functions of the design variables. The stress constraints can be any function of deflections, and so can the deflection constraints. The actual form of such constraints only modifies the feasible region in the space of the deflections in sub-problem (3), while leaving unaltered the form of sub-problem (4) on which the theorem depends.

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Абстракт — Исследуется задача расчета на минимум веса упругих конструкций, при условиях многократной нагрузки. Оказывается, что задача выражается в качестве исследования возможных форм прогиба, сопряженного с повторными поисками конструкций на минимум веса, для заданной жесткости. Последняя задача является задачей линейного программирования и заключает в себе верхний предел, отличный от этого, построенного посредством связности, относительно и числа и плотности распределения элементов, присутствующих в расчете на минимум веса.